MATH0076 Algebraic Geometry, Sheet 1

Ed Segal

1. Which of the following are affine varieties? Justify your answers.
   (a) $\{(t^2, t^3),\ t \in \mathbb{C}\} \subset \mathbb{A}^2$.
   (b) The single point $(\pi, e^\pi) \in \mathbb{A}^2$.
   (c) $\{|z|^2 = 1\} \subset \mathbb{A}^1$.
   (d) $\{(z, w),\ |z|^2 + |w|^2 = 1\} \subset \mathbb{A}^2$.
   (e) $\{(z, w),\ \sin z = 0,\ \cos w = 3,\ |z|^2 + |w|^2 < 100\} \subset \mathbb{A}^2$.
   (f) The group $SL_2(\mathbb{C})$ of 2×2 complex matrices of determinant 1.

$\star$2$\star$. (a) Let $V_1 = V(f_1, ..., f_k)$ and $V_2 = V(g_1, ..., g_m)$ be two affine varieties in $\mathbb{A}^n$. Prove that $V_1 \cup V_2$ is an affine variety by finding a finite list of polynomials that define it.
   (b) Give an example of a countable collection of affine varieties whose union is not an affine variety.

3. Let I be an ideal in a ring R. Show that the radical

$$\mathrm{rad}(I) = \left\{ f \in R,\ f^k \in I \text{ for some } k \right\}$$

   is an ideal.

4. Let $V_1, V_2 \subset \mathbb{A}^n$ be two affine varieties, and let $I_1 = I_{V_1}$ and $I_2 = I_{V_2}$ be their associated radical ideals in $\mathbb{C}[x_1, ..., x_n]$.
   (a) Show that $I(V_1 \cup V_2) = I_1 \cap I_2$. How does this compare to your answer for Q2(a)?

   The *sum* of I_1 and I_2, denoted $I_1 + I_2$, is the ideal generated by $I_1 \cup I_2$.

   (b) Show that $V(I_1 + I_2) = V_1 \cap V_2$.
   (c) It is not always true that $I_{V_1 \cap V_2} = I_1 + I_2$. Find a counter-example.

5. A $\mathbb{C}$-*algebra* is a ring R containing $\mathbb{C}$ as a subring. A $\mathbb{C}$-algebra is *finitely-generated* if there is a finite set of elements $r_1, ..., r_n \in R$ such the set $\mathbb{C} \cup \{r_1, ..., r_n\}$ generates R as a ring.
   (a) Show that every finitely-generated $\mathbb{C}$-algebra is isomorphic to $\mathbb{C}[x_1, ..., x_n]/I$ for some n and some ideal I.

   An element r of a ring R is *nilpotent* if $r^k = 0$ for some k, and a ring is called *reduced* if it contains no nilpotent elements (except for 0).

   (b) Show that a ring R is the co-ordinate ring of a (complex) affine variety if and only R is a finitely-generated reduced $\mathbb{C}$-algebra.

MATH 0076 Algebraic Geometry, Sheet 2

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1. (i) Let $V_1, V_2 \subset \mathbb{A}^n$ be two affine varieties. Explain why there is an injective ring homomorphism:

$$\rho : \mathbb{C}[V_1 \cup V_2] \longrightarrow \mathbb{C}[V_1] \oplus \mathbb{C}[V_2]$$

   Show that ρ is not surjective if $V_1 \cap V_2 \neq \phi$.

   (ii) Consider the *real* affine varieties $V_1 = V(y - x^2), V_2 = V(y + 1)$ in $\mathbb{A}^2(\mathbb{R})$. Show that ρ is not surjective, even though $V_1 \cap V_2 = \phi$.

   (iii) Now prove that for any two complex affine varieties $V_1, V_2 \subset \mathbb{A}^n$, if $V_1 \cap V_2 = \phi$ then ρ is an isomorphism.

2. Let $V = V(y) \subset \mathbb{A}^2$ and $W = V(y - x^2) \subset \mathbb{A}^2$. Write down a function $F : V \cup W \to \mathbb{C}$ such that $F|_V$ is regular and $F|_W$ is regular but F is not regular.

3. (a) Two possible definitions of a *Noetherian* ring R are:

   (i) Every idea in R is finitely-generated.

   (ii) R does not contain an infinite strictly increasing chain of ideals:

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq ...$$

   In the lecture notes it's proved that (i) implies (ii). Prove that (ii) implies (i).

   (b) Prove that $\mathbb{C}[V]$ is Noetherian for any affine variety V.

4. Prove that the decomposition of an affine variety into irreducible components is unique.

5. Is the intersection of two irreducible affine varieties always irreducible? Give a proof or counter-example.

$\star$6$\star$. Find the irreducible components of the following varieties:

   (a) $V(xyz) \subset \mathbb{A}^3$

   (b) $V(xy,\, yz,\, xz) \subset \mathbb{A}^3$

   (c) $V(x^4 - x^2 - 2) \subset \mathbb{A}^1$

7. Find the irreducible components of $V(y^2 - xy - x^2y + x^3) \subset \mathbb{A}^2$.

MATH 0076 Algebraic Geometry, Sheet 3

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1. (a) Let $V = V(x^2 + y^2 + z) \subset \mathbb{A}^3$. Show that V is isomorphic to $\mathbb{A}^2$.
   (b) Let $V = V(y^2 - y) \subset \mathbb{A}^2$ and $W = V(xy) \subset \mathbb{A}^2$. Construct a surjective regular map $F : V \to W$. Are V and W isomorphic?

$\star$2$\star$. Let $F : V \to W$ be a regular map between two affine varieties, and let $F^* : \mathbb{C}[W] \to \mathbb{C}[V]$ be the corresponding map between their co-ordinate rings.
   (a) Prove that if F is surjective then F^* is injective.
   (b) If $p, q \in V$ are two distinct points, show that there is a regular function $f \in \mathbb{C}[V]$ such that $f(p) \neq f(q)$. Now prove that if F^* is surjective then F must be injective.

3. (a) Is the converse to Q2(a) true? Is the converse to Q2(b) true?
   (b) Use Q2(a) to prove that if $F : V \to W$ is a surjection, and V is irreducible, then W must be irreducible.

4. A function $F : \mathbb{A}^n \to \mathbb{A}^m$ is called *affine* if it is the composition of a linear map and a translation, *i.e.* if $F : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ for some matrix A and m-vector $\mathbf{b}$.
   (a) Prove that every isomorphism $F : \mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$ is affine.
   (b) Find an isomorphism $F : \mathbb{A}^2 \xrightarrow{\sim} \mathbb{A}^2$ which is not affine.

5. Prove that $\mathbb{A}^1$ is not isomorphic to $\mathbb{A}^2$.

6. (a) Let $V = V(t^2 - t) \subset \mathbb{A}^1$ and $W = V(x - y, xy - 1) \subset \mathbb{A}^2$. Write down an isomorphism between V and W.
   (b) Now let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^k$ be two finite sets of the same size. Prove that V and W are isomorphic affine varieties. *You may find Q1 on Sheet 2 helpful.*

MATH 0076 Algebraic Geometry, Sheet 4

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1. Let V be an irreducible affine variety. Prove that:

   (a) If $U_1, U_2 \subset V$ are any two non-empty Zariski open subsets then the intersection $U_1 \cap U_2$ is not empty.

   (b) If $U \subset V$ is any non-empty Zariski open subset then the Zariski closure of U is V.

   Now prove that if V is reducible both (a) and (b) are false.

2. (a) Let $S \subset \mathbb{A}^n$ be any subset, and let $I(S) \subset \mathbb{C}[x_1, ..., x_n]$ be the ideal of polynomials vanishing on S. Show that $V(I(S))$ is the Zariski closure of S.

   (b) Find the Zariski closure of $S = \{(n, n), n \in \mathbb{Z}\} \subset \mathbb{A}^2$.

3. For each of the following rational functions in $\mathbb{C}(x, y)$ find their set of regular points.

   (a) $\dfrac{y}{x^3 - x^2 y}$ (b) $\dfrac{x+1}{x^2 + xy + x + y}$

4. Let f be a polynomial in n variables and let U be the quasi-affine variety $U = \mathbb{A}^n \setminus V(f)$.

   $\star$(a)$\star$ Show that U is isomorphic to an affine variety in $\mathbb{A}^{n+1}$.

   (b) Show that a rational function g/h is regular on U if and only if every irreducible factor of h divides f.

5. (a) Let $U \subset \mathbb{A}^n$ be Zariski open and let $F : U \to \mathbb{A}^k$ be a regular function. Show that if $W \subset \mathbb{A}^k$ is Zariski open then the pre-image $F^{-1}(W) \subset U$ is Zariski open.

   (b) Let $F : U \to U'$ and $G : U' \to U''$ be regular functions between any quasi-affine varieties U, U', U''. Prove that $G \circ F$ is regular.

6. Let $V = V(y^2 - x^3)$ and let U be the quasi-affine variety $V \setminus (0, 0)$. Show that U is isomorphic to $\mathbb{A}^1 \setminus 0$.

7. *This continues Sheet 3 Q1(b).* Prove that the following two quasi-affine varieties in $\mathbb{A}^2$ are isomorphic:

$$U = V(y^2 - y) \setminus V(x) \qquad U' = V(xy) \setminus (0, 0)$$

MATH 0076 Algebraic Geometry, Sheet 5

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1. (a) Let $F : U \xrightarrow{\sim} W$ is an isomorphism between two quasi-affine varieties. Prove that F induces an isomorphism between the rings of regular functions on U and W.

   (b) Let $U = \mathbb{A}^2 \setminus (0,0)$, let W be an affine variety, and let $F : U \to W$ be a regular map. Show that F must be the restriction of some regular map $\hat{F} : \mathbb{A}^2 \to W$. Now use part (a) to prove that F cannot be an isomorphism.

   Hint: the restriction map from $\mathbb{C}[\mathbb{A}^2]$ to regular functions on U is an isomorphism.

2. (a) Let $V = V(y^2 - x^3)$. Find an isomorphism between $\mathbb{C}(V)$ and $\mathbb{C}(t)$.

   (b) Repeat (a) for $V = V(y^2 - x^k)$ where $k \in \mathbb{N}$ is an odd number.

   (c) What happens if k is even?

$\star$3.$\star$ Let $V = V(y - x^2) \subset \mathbb{A}^2$. For each of the following two rational functions in $\mathbb{C}(x, y)$, say if it defines a rational function on V and if it does find the set of regular points.

$$\text{(a)} \quad \frac{y}{xy - x^3} \qquad\qquad \text{(b)} \quad \frac{1}{y - x}$$

4. Let $V = V(xy - z^2) \subset \mathbb{A}^3$ and let $\psi = y/z \in \mathbb{C}(V)$.

   (a) Show that $(1, 0, 0)$ is a regular point of ψ.

   (b) Guess which points are not regular.

   (c) Prove your guess is correct.

5. Let $V = V(y^2 - x^3)$ and let $\psi = x/y \in \mathbb{C}(V)$. Prove that $(0, 0)$ is not a regular point of ψ.

MATH 0076 Algebraic Geometry, Sheet 6

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$\star$1$\star$. Define $F : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ by $F(x,y) = \left(\frac{x}{xy-1}, \frac{xy-1}{x}\right)$.

(a) What are the regular points of F? What is the image of F?

(b) Does $F \circ F$ make sense?

2. Let $V = V(xy - zw) \subset \mathbb{A}^4$ and consider the rational map $F : V \dashrightarrow \mathbb{A}^2$ given by:

$$(x, y, z, w) \mapsto (s, t) = \left(\frac{w}{y}, \frac{w}{z}\right)$$

(a) Find the regular points of F.

(b) Show that F is dominant.

(c) Let $\phi = s/t \in \mathbb{C}(s,t)$.

(i) If $p \in V$ is a regular point of $F^*(\phi)$, does it follow that p is a regular point of F?

(ii) If p is a regular point of $F^*(\phi)$ and F, does it follow that $F(p)$ is a regular point of ϕ?

3. Define $F : \mathbb{A}^2 \to \mathbb{A}^2$ by $F(x,y) = (xy, xy^2)$. Show that F is a birational equivalence.

4. Prove Lemma 11.15 in the notes.

5. Let $V = V(xy - z^2) \subset \mathbb{A}^3$.

(a) Let $F : \mathbb{A}^2 \to V$ be the map $F(s,t) = (s^2, t^2, st)$. Is F a birational equivalence?

(b) Find a birational equivalence between $\mathbb{A}^2$ and V.

MATH 0076 Algebraic Geometry, Sheet 7

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$\star$1$\star$. (a) Let $V \subset \mathbb{A}^n$ be an affine variety with $I_V = \langle f_1, .., f_k \rangle$, and fix a point $p \in V$. For $v \in \mathbb{C}^n$, show that:

$$(Df_i)_p(v) = 0,\ \forall i \in [1,k] \quad \iff \quad Dg_p(v) = 0,\ \forall g \in I_V$$

(b) Let $f = y - x^2$ and $g = y$, let $V = V(f,g) \subset \mathbb{A}^2$, and let p be the origin.

(i) Find the subspace:

$$\operatorname{Ker} Df_p \cap \operatorname{Ker} Dg_p \ \subset \mathbb{C}^2$$

(ii) What is T_pV?

2. For each of the following varieties, find the dimension of T_pV for each point $p \in V$. Hence identify the singular points and state the dimension of V.

(a) $V = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$.
(b) $V = V(z - \frac{1}{2}x^2 - \frac{1}{2}y^2,\ z - x - y) \subset \mathbb{A}^3$.
(c) $V = V(x^2 + ty^2,\ s^3 - t^2) \subset \mathbb{A}^4$.

(In each case you may assume that V is irreducible and that the given polynomials generate $I(V)$.)

3. (a) Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a polynomial and let $\partial_i f$ denote its partial derivatives. Show that $\partial_i f$ doesn't lie in the ideal $\langle f \rangle$ unless $\partial_i f = 0$.

(b) Let $V \subset \mathbb{A}^n$ be an irreducible hypersurface. Prove that $\dim V = n-1$.

4. (a) Let V be an irreducible affine variety. Show that the two possible definitions of a *non-singular point* of V agree.

(b) Find the singular points of:

(i) $V(xz, yz, xw, yw) \subset \mathbb{A}^4$
(ii) $V(y - x^2,\ z - x^3) \cup V(y + z) \subset \mathbb{A}^3$

5. Identify $\mathbb{A}^6$ with the set of 2×3 matrices, and let $V \subset \mathbb{A}^6$ be the subset of matrices of rank ≤ 1.

(a) Show that V is an affine variety.

(b) Compute the field $\mathbb{C}(V)$ and deduce the dimension of V. *(No proofs required.)*

MATH 0076 Algebraic Geometry, Sheet 8

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$\star 1\star$. (a) Suppose a point $p \in \mathbb{P}^2$ maps to $(x, y) \in \mathbb{A}^2$ under the first standard chart. Find the image of p under the other two standard charts. Are there any values of (x, y) for which this doesn't make sense, and if so why not?

(b) Let

$$\mathbb{V} = \mathbb{V}(x - y) = \{x\!:\!y\!:\!z,\ x = y\} \ \subset \mathbb{P}^2$$

What does $\mathbb{V}$ look like? What does $\mathbb{P}^2 \setminus \mathbb{V}$ look like? *(No proofs required!)*

2. (a) Show that an invertible square matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ induces a function:

$$\widehat{A} : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}$$

Why does A have to be invertible? For which matrices is $\widehat{A}$ the identity function?

(b) For any three distinct points $p_0, p_1, p_2 \in \mathbb{P}^1$ show that there is a function $\widehat{A}$ as in part (a) such that:

$$\widehat{A}(p_0) = 0\!:\!1, \qquad \widehat{A}(p_1) = 1\!:\!0, \qquad \widehat{A}(p_2) = 1\!:\!1$$

3. For each of the following projective varieties in $\mathbb{P}^2$ write down the three corresponding affine varieties in $\mathbb{A}^2$ that we get by intersecting with the three standard charts. What does each projective variety look like?

(a) $\mathbb{V}(xyz)$

(b) $\mathbb{V}(y^2 z - x^3)$

(c) $\mathbb{V}(x^2 y - xz^2,\ xy^2 - yz^2)$

4. For the following affine hypersurfaces in $\mathbb{A}^n$ write down their projective completion in $\mathbb{P}^n$. In each case give a brief description of the set of additional points in the $\mathbb{P}^{n-1}$ 'at infinity'.

(a) $V(x^2 - y^2 - 1) \subset \mathbb{A}^2$

(b) $V(x^2 + y^2 + z^2 + xyz + 1) \subset \mathbb{A}^3$

(c) $V(xy - z^2) \subset \mathbb{A}^3$

5. (a) Show that the union of two projective varieties is a projective variety.

(b) Find a countable set of projective varieties whose union is not a projective variety.

(c) Show that the intersection of any set of projective varieties is a projective variety. *(Hint: affine cones.)*

6. (a) Let $\mathbb{V} = \mathbb{V}(x^2 z + yz^2) \subset \mathbb{P}^2$, and let $V \subset \mathbb{A}^2$ be the intersection of $\mathbb{V}$ with the standard chart $\{z \neq 0\}$. Is $\mathbb{V}$ the projective completion of V?

(b) What do you think the projective completion of

$$V(xy + z,\ xy - z) \subset \mathbb{A}^3$$

is? Can you see any ambiguity here? Can you find a definition of 'projective completion' that doesn't rely on choosing polynomials?

MATH 0076 Algebraic Geometry, Sheet 9

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1. (a) Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety and let $V \subset \mathbb{A}^n$ be the intersection of V with a standard chart. Prove that if $\mathbb{V}$ is irreducible then V is irreducible.

   (b) Find a projective variety $\mathbb{V} \subset \mathbb{P}^1$ such that $\mathbb{V}$ is reducible, but the intersection of $\mathbb{V}$ with either standard chart gives an irreducible affine variety. Repeat for $\mathbb{V} \subset \mathbb{P}^n$.

2. Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety and let $V \subset \mathbb{A}^n$ be the intersection of V with one of the standard charts. Show that a subset $U \subset V$ is Zariski open iff $U = \mathbb{U} \cap V$ for some Zariski open subset $\mathbb{U} \subset \mathbb{V}$.

3. For each of the following projective varieties find the set of singular points. Say if they are irreducible *(no proofs required)*, and if so state their dimension.

   (a) $\mathbb{V}(xz, yz) \subset \mathbb{P}^2$

   (b) $\mathbb{V}(y^2z + x^3 + z^3) \subset \mathbb{P}^2$

   (c) $\mathbb{V}(xyz - w^3) \subset \mathbb{P}^3$

4. Let $V \subset \mathbb{A}^{n+1}$ be a cone and let $\mathbb{V} \subset \mathbb{P}^n$ be the corresponding projective variety. Suppose $\mathbb{V}$ has no singular points. Does it follow that V has no singular points? If not what is the correct statement?

5. (a) Let $\Psi : \mathbb{A}^n \dashrightarrow \mathbb{P}^2$ be a rational map. Show that from Ψ we can construct three rational maps $\Phi_1, \Phi_2, \Phi_3 : \mathbb{A}^n \dashrightarrow \mathbb{A}^2$, and that a point is regular for Ψ iff it is regular for at least one of the Φ_i.

   (b) Show that $(f_0, ..., f_k)$ and $(g_0, ..., g_k)$ define the same rational map $\mathbb{A}^n \dashrightarrow \mathbb{P}^k$ if and only if we have $f_i g_j = f_j g_i$ for all $i, j \in [0, k]$.

$\star$6$\star$. Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map $\Phi = xy : yz : xz$.

   (a) Find the base-points of Φ.

   (b) Describe the behaviour of Φ on the subvariety $\mathbb{V}(xyz) \subset \mathbb{P}^2$.

7. (a) Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a regular map, and assume $1 : 0$ is not in the image of F. Show that F must be constant.

   (b) How would you define a *regular map* from $\mathbb{P}^1$ to an affine variety? Now show that any such map must by constant.

   (c) Let $U = \mathbb{P}^2 \setminus 0 : 0 : 1$. Sketch a proof that U is not isomorphic to any affine variety.

MATH 0076 Algebraic Geometry, Sheet 10

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1. Let
$$\mathbb{V} = \mathbb{V}(xz - y^2,\, yw - z^2,\, xw - yz) \ \subset \mathbb{P}^3$$
be the twisted cubic curve.
   (a) Prove that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$. *(Follow Example 17.19 in the notes.)*
   (b) Now express your isomorphism $\mathbb{V} \xrightarrow{\sim} \mathbb{P}^1$ in terms of the function field $\mathbb{C}(V)$, where V is the affine cone of $\mathbb{V}$.

2. Let $\mathbb{V} = \mathbb{V}(f) \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the hypersurface defined by:
$$f(x, y, s, t) = x^2 t - y^2 s$$
Prove that $\mathbb{V}$ is isomorphic to $\mathbb{P}^1$.

3. (a) Let $f \in \mathbb{C}[x, y]$ be any homogeneous polynomial. Describe $\mathbb{V}(xf, yf) \subset \mathbb{P}^1$.
   (b) Consider the following hypersurface of bidegree (2,3):
$$\mathbb{V} = \mathbb{V}(x^2 s^3 + y^2 t^3) \ \subset \mathbb{P}^1 \times \mathbb{P}^1$$
   Write down a projective variety in $\mathbb{P}^3$ which is isomorphic to $\mathbb{V}$. *(No proofs required.)*

4. Let $\mathbb{B} = \mathbb{V}(xt - ys) \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the blow-up of $\mathbb{A}^2$ at the origin, and let $\pi : \mathbb{B} \to \mathbb{A}^2$ be the map:
$$\pi : (x, y, s\!:\!t) \mapsto (x, y)$$
   (a) Show that π is a birational equivalence.
   (b) Let $\Psi : \mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ be the rational map $\Psi = x^2 : (y^2 - x^5)$. Find a regular map $F : \mathbb{B} \to \mathbb{P}^1$ which agrees with Ψ on the set $\mathbb{B} \setminus \mathbb{V}(x, y)$.
   (c) Try part (b) again but set $\Phi = x^3 : (y^2 - x^5)$ instead. Can you still do it?

5. Let $\pi : \mathbb{B} \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin, and for a hypersurface $V \subset \mathbb{A}^2$ let $\mathbb{W} \subset \mathbb{B}$ denote the proper transform of V. For the following examples, find equations for $\mathbb{W}$ in $\mathbb{A}^2 \times \mathbb{P}^1$ and find the subset $\pi^{-1}(0,0) \cap \mathbb{W}$.
   (a) $V = V\big(xy(x + y)\big)$
   (b) $V = V\big(y^2 - x^3 - x^2\big)$

6. Let

$$\mathbb{B} = \mathbb{V}(xt - ys,\ xu - zs,\ yu - zt) \subset \mathbb{A}^3 \times \mathbb{P}^2$$

be the blow-up of $\mathbb{A}^3$ at the origin, and let $\pi : \mathbb{B} \to \mathbb{A}^3$ be the projection:

$$\pi : (x, y, z, s : t : u) \mapsto (x, y, z)$$

Now let $V = V(xy - z^2) \subset \mathbb{A}^2$. By looking in charts verify that

$$\pi^{-1}(V) = \mathbb{V}(x, y, z) \cup \left(\mathbb{B} \cap \mathbb{V}(st - u^2)\right)$$

and check that the proper transform of V has no singular points.